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Conservation laws and transverse motion of energy on reflection and transmission of electromagnetic waves

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Abstract. The conservation laws for the process of reflection and transmission of electromagnetic waves on a plane interface of isotropic transparent media are determined. Using these laws, relations have been established between the transverse shift (TS) of a centre of gravity of reflected and transmitted wavepackets, the change of the normal component of the intrinsic Minkowski angular momentum of the electromagnetic field and the Abraham transverse momentum (or the transverse electromagnetic power flow (TPF)). The previous investigations of the TS and TPF phenomena are discussed from the point of view of conservation laws.

1. Introduction

The transverse shift (TS) of a totally reflected light beam was predicted three decades ago (Fedorov 1955, Kristoffel 1956). Since that time a number of theoretical papers devoted to the TS phenomenon have been published. There are different opinions both about the conditions of the existence of this effect and its value.

From the physical point of view the TS is a consequence of a transverse power flow (TPF). For a long time the TPF has been assumed to be associated with Fresnel's evanescent waves or, more generally, with inhomogeneous waves (Wiegrefe 1914, Fedorov 1955, 1977, Kristoffel 1956, Imbert 1972). On the basis of this assumption a method of calculation of the TS of a totally reflected light beam has been developed (Kristoffel 1956, Imbert 1972, Fedorov 1977). This method is usually called the 'energy-flux method'.

Schilling (1965) has calculated the TS of a reflected light beam using the stationary-phase method. In the case of total reflection, his result numerically differs from that obtained by the energy-flux method. It also follows from Schilling's paper that the elliptically polarised light beam should undergo a TS not only in the case of total but also of partial reflection. However, Schilling does not give the physical reason for the TS of a partially reflected light beam.

De Beauregard (1965) has considered the TS of the totally reflected light beam as a 'translational inertial spin effect'. The value of the TS calculated by him is different from the values obtained by the 'energy-flux method' and the stationary-phase method.

In an earlier paper of the author (Fedoseyev 1985) the TS of a transmitted elliptically polarised light beam was predicted. In this paper, the TS has been considered as a change of the transverse coordinate of the wavebeam's centre of gravity on reflection and refraction. For total reflection such a definition of TS has been used by Boulware (1973).

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The TS of a light beam on total reflection was first observed by Imbert (1972). Other experimental investigations of the TS of the totally reflected electromagnetic wavebeams were later performed (de Beauregard *et al* 1977, Pun'ko and Filippov 1984, 1985).

In connection with Imbert's experiments it was discussed which of the two methods, either energy-flux or stationary-phase, agrees with the experimental value of the TS of a totally reflected light beam (Imbert 1972, Ashby and Miller 1973, 1977, Boulware 1973, de Beauregard *et al* 1977). Imbert (1972) found that the experimental value of the TS agreed with that calculated by the energy-flux method, but a more detailed analysis of his experimental conditions showed that the experimental value of the TS agreed with that calculated by the stationary-phase method (Ashby and Miller 1973, Boulware 1973). In the course of the discussions it was not established why the energy-flux and stationary-phase methods give different results for the TS of a totally reflected light beam.

In the present paper, we investigate the TS and TPF phenomena in a general way: from the point of view of the conservation laws valid in the case of reflection and refraction of electromagnetic waves on the plane interface of two isotropic transparent dielectric media. Based on the conservation laws, we will obtain the relations between the TS of the centre of gravity of the electromagnetic field, the TPF and the change of an intrinsic angular momentum of the electromagnetic field.

Our analysis allows for the electromagnetic field in both media. In this respect our approach differs from that of de Beauregard *et al* (1971) who analysed energy-momentum quanta in Fresnel's evanescent wave without properly considering the electromagnetic field in a denser medium.

2. Conservation laws

First we consider the conservation laws for an electromagnetic field embedded in the space filled with two half-infinite immobile isotropic transparent dielectric media separated by a plane interface. $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ are the relative dielectric permittivities of the first and the second media and $n^{(1,2)}=(\varepsilon^{(1,2)})^{1/2}$ are the refraction indices of the media. For simplicity, the dispersion of dielectric permittivities is not taken into account and the relative magnetic permeability in both media is taken equal to unity.

Denote by $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{H}(\boldsymbol{x}, t)$ the electric and magnetic field vectors at an arbitrary space point $\boldsymbol{x}=(x_1, x_2, x_3)$ and at an arbitrary instant t. Suppose that $|\boldsymbol{E}(\boldsymbol{x}, t)|$ and $|\boldsymbol{H}(\boldsymbol{x}, t)|$ decrease faster than $|\boldsymbol{x}|^{-3/2}$ if $|\boldsymbol{x}|\to\infty$.

2.1. *Energy and momentum*

The energy and momentum conservation laws are written in a differential form as follows (Møller 1972, Ginsburg 1973):

$$\frac{\partial T_{ik}(\boldsymbol{x}, t)}{\partial x_k}=0 \tag{1}$$

where $T_{ik}(\boldsymbol{x}, t)$ is the Minkowski energy-momentum tensor

$$T_{ik}(\boldsymbol{x}, t)=\begin{pmatrix}\sigma_{\alpha\beta}(\boldsymbol{x}, t) & -\mathrm{i}g(\boldsymbol{x}, t)\\ -\mathrm{i}s(\boldsymbol{x}, t) & w(\boldsymbol{x}, t)\end{pmatrix} \tag{2}$$

and $i, k = 1, 2, 3, 4$; $\alpha, \beta = 1, 2, 3$; $x_4 = \mathrm{i}t$. $\boldsymbol{s}(\boldsymbol{x}, t)$ is the Poynting vector

$$\boldsymbol{s}(\boldsymbol{x}, t) = (1/4\pi)\boldsymbol{E}(\boldsymbol{x}, t) \times \boldsymbol{H}(\boldsymbol{x}, t). \tag{3}$$

The light velocity in vacuum is taken equal to unity, hence $\boldsymbol{s}(\boldsymbol{x}, t)$ coincides with the Abraham momentum density. $\boldsymbol{g}(\boldsymbol{x}, t)$ is the Minkowski momentum density

$$\boldsymbol{g}(\boldsymbol{x}, t) = \tilde{\varepsilon}\boldsymbol{s}(\boldsymbol{x}, t) \tag{4}$$

where $\tilde{\varepsilon} = \varepsilon^{(1)}$ in the first medium and $\tilde{\varepsilon} = \varepsilon^{(2)}$ in the second medium. $w(\boldsymbol{x}, t)$ is the electromagnetic energy density

$$w(\boldsymbol{x}, t) = (1/8\pi)(\tilde{\varepsilon}|\boldsymbol{E}(\boldsymbol{x}, t)|^2 + |\boldsymbol{H}(\boldsymbol{x}, t)|^2). \tag{5}$$

$\sigma_{\alpha\beta}(\boldsymbol{x}, t)$ is the Maxwellian stress tensor:

$$\sigma_{\alpha\beta}(\boldsymbol{x}, t) = (1/4\pi)(\tilde{\varepsilon}E_\alpha(\boldsymbol{x}, t)E_\beta(\boldsymbol{x}, t) + H_\alpha(\boldsymbol{x}, t)H_\beta(\boldsymbol{x}, t)) - \delta_{\alpha\beta}w(\boldsymbol{x}, t) \tag{6}$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol.

Let us direct the x_1 axis perpendicular to the interface. $\sigma_{21}(\boldsymbol{x}, t)$ and $\sigma_{31}(\boldsymbol{x}, t)$ as well as $s_1(\boldsymbol{x}, t)$ are continuous at the interface. Hence, integrating (1) over total space for $i = 2, 3, 4$ we obtain three conservation laws:

$$\partial W(t)/\partial t = 0 \tag{7a}$$

$$\partial G_{2,3}(t)/\partial t = 0 \tag{7b}$$

where $W(t)$ is the electromagnetic energy

$$W(t) \equiv W = \int w(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x} \tag{8}$$

and $\boldsymbol{G}(t)$ is the Minkowski momentum

$$\boldsymbol{G}(t) = \int \boldsymbol{g}(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x}. \tag{9}$$

Let us direct the x_2 axis perpendicular to the vector $\boldsymbol{G}(t)$ at the initial instant of time. Then at an arbitrary time

$$G_2(t) = 0 \tag{9a}$$

$$G_3(t) \equiv G_3 = \text{constant}. \tag{9b}$$

Now consider the Abraham momentum

$$\boldsymbol{S}(t) = \int \boldsymbol{s}(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x}. \tag{10}$$

One can express $\boldsymbol{S}(t)$ as $\boldsymbol{S}(t) = \boldsymbol{S}^{(1)}(t) + \boldsymbol{S}^{(2)}(t)$, where $\boldsymbol{S}^{(1)}(t)$ and $\boldsymbol{S}^{(2)}(t)$ are the Abraham momenta in the first and second media, respectively. $\boldsymbol{S}^{(1,2)}(t)$ are defined by (10) when the integration is performed over the volume of the first or second media, respectively. By using (4) and (9a) one obtains

$$\varepsilon^{(1)}S_2^{(1)}(t) + \varepsilon^{(2)}S_2^{(2)}(t) = 0 \tag{11}$$

and hence

$$S_2^{(1)}(t) = (1 - \varepsilon^{-1})^{-1}S_2(t) \tag{12a}$$

$$S_2^{(2)}(t) = (1 - \varepsilon)^{-1}S_2(t) \tag{12b}$$

where $\varepsilon = \varepsilon^{(2)}(\varepsilon^{(1)})^{-1}$.

2.2. *Angular momentum*

Let us consider the angular momentum 4-tensor (Møller 1972)

$$\mathcal{M}_{ik}(t) = \mathrm{i} \int (x_i T_{k4}(\boldsymbol{x}, t) - x_k T_{i4}(\boldsymbol{x}, t))\, \mathrm{d}\boldsymbol{x}. \tag{13}$$

Differentiating (13) for $i = 2$, $k = 3$ and taking into account (1) as well as the continuity of $\sigma_{21}(\boldsymbol{x}, t)$ and $\sigma_{31}(\boldsymbol{x}, t)$ at the interface we obtain

$$\partial M_1(t)/\partial t = 0 \tag{14}$$

where $M_1(t) = \mathcal{M}_{23}(t)$. $M_1(t)$ is the normal, with respect to the interface, component of the angular momentum of the electromagnetic field

$$\boldsymbol{M}(t) = \int \boldsymbol{x} \times \boldsymbol{g}(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x}. \tag{13a}$$

(Throughout this paper only the Minkowski angular momentum is considered.)

Let us split $\boldsymbol{M}(t)$ into two parts:

$$\boldsymbol{M}(t) = \boldsymbol{L}(t) + \boldsymbol{I}(t). \tag{13b}$$

$\boldsymbol{L}(t)$ is the orbital angular momentum (or the angular momentum of the centre of gravity of the electromagnetic field)

$$\boldsymbol{L}(t) = \boldsymbol{X}(t) \times \boldsymbol{G}(t) \tag{15}$$

where $\boldsymbol{X}(t)$ is the radius vector of the centre of gravity of the electromagnetic field

$$\boldsymbol{X}(t) = W^{-1} \int \boldsymbol{x} w(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x} \tag{16}$$

and $\boldsymbol{I}(t)$ is the intrinsic angular momentum of the electromagnetic field

$$\boldsymbol{I}(t) = \int (\boldsymbol{x} - \boldsymbol{X}(t)) \times \boldsymbol{g}(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x}. \tag{17}$$

Substituting (15) into (13b) and the result into (14) and making use of (7b) we obtain

$$\partial X_2(t)/\partial t = -(1/G_3)\partial I_1(t)/\partial t. \tag{18}$$

2.3. TS *of the electromagnetic field*

Let us define the TS of the electromagnetic field for a time interval $t_2 - t_1$ as follows:

$$h = X_2(t_2) - X_2(t_1). \tag{19}$$

Integrating (18) we obtain the relation between h and the change of the normal component of the intrinsic angular momentum of the electromagnetic field for the time interval $t_2 - t_1$:

$$h = -G_3^{-1} \Delta I_1 \tag{20}$$

where

$$\Delta I_1 = I_1(t_2) - I_1(t_1). \tag{21}$$

On the basis of differential conservation law (1) for $i=4$, one can also obtain a relation between the TS of the electromagnetic field and the integral Abraham momentum. Let us differentiate (16). On the right-hand side of the result use the equation

$$\frac{\partial}{\partial t} w(\boldsymbol{x}, t) = -\frac{\partial}{\partial x_\beta} s_\beta(\boldsymbol{x}, t)$$

(i.e. (1) for $i=4$), and after that perform the integration by parts, taking into account that $s_1(\boldsymbol{x}, t)$ is continuous at the interface. Then we get

$$\partial X_2(t)/\partial t = W^{-1} S_2(t). \qquad (22)$$

Integrating (22), we obtain

$$h = W^{-1} \int_{t_1}^{t_2} S_2(t)\, \mathrm{d}t. \qquad (22a)$$

3. Reflection and transmission of an electromagnetic wavepacket

Let us now consider the process of reflection and transmission of a time- and space-restricted quasimonochromatic packet of electromagnetic waves (pulse of waves). Suppose that the packet impinges upon the interface from the first medium (the upper medium in figure 1). Assume that the incident packet is far enough from the interface at the initial instant of time t_1 (figure 1(a)), such that one can neglect the influence of the second medium on the incident packet at this instant of time. Analogously, assuming that the reflected and transmitted packets are far enough from the interface at the instant of time t_2, one can neglect the influence of the second medium on the reflected packet and the influence of the first medium on the transmitted one at this instant of time (figure 1(c)). At the intermediate time interval three packets coexist (figure 1(b)).

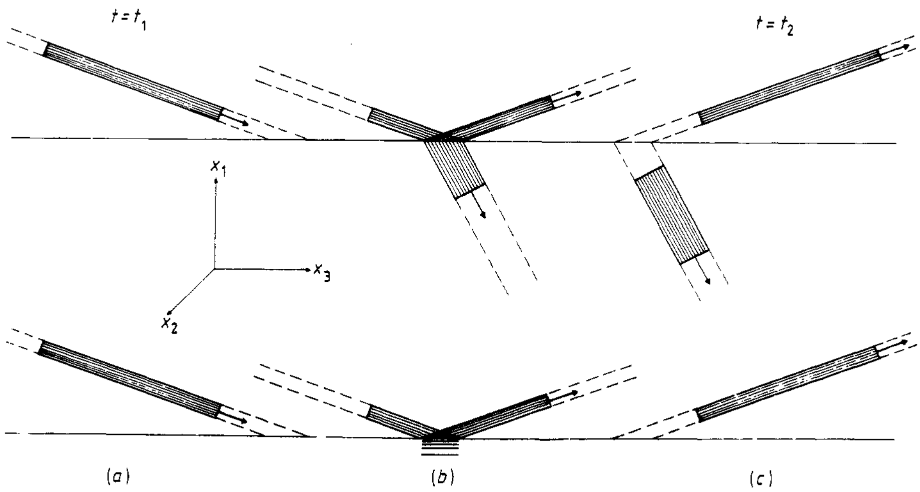

Figure 1. The scheme of the partial reflection and transmission (upper) and of the total reflection (lower) of the time-restricted wavepacket (full lines) and of the wavebeam (broken lines).

The field quantities of the incident packet will be denoted by an upper index (i) and the field quantities of the reflected and transmitted packets by upper indices (ρ) and (τ). Further we will use the common notation t_j for t_1 and t_2 as follows: $t_j = t_1$ if $j = i$ and $t_j = t_2$ if $j = \rho$ or τ.

In the appendix, definitions of some field quantities for the jth wavepacket at the instant t_j are given and calculations of some quantities are presented.

Each wavepacket is characterised by three orthogonal unit vectors $\boldsymbol{a}$, $\boldsymbol{m}^{(j)}$ and $\boldsymbol{b}^{(j)}$ (figure 2), where $\boldsymbol{a}$ is the unit vector directed along the x_2 axis,

$$\boldsymbol{m}^{(j)} = |\boldsymbol{G}^{(j)}(t_j)|^{-1}\boldsymbol{G}^{(j)}(t_j) \tag{23a}$$

and

$$\boldsymbol{b}^{(j)} = \boldsymbol{a} \times \boldsymbol{m}^{(j)}. \tag{23b}$$

The angle between the x_1 axis and the vector $\boldsymbol{m}^{(i)}$ is the angle of incidence, θ (see figure 2).

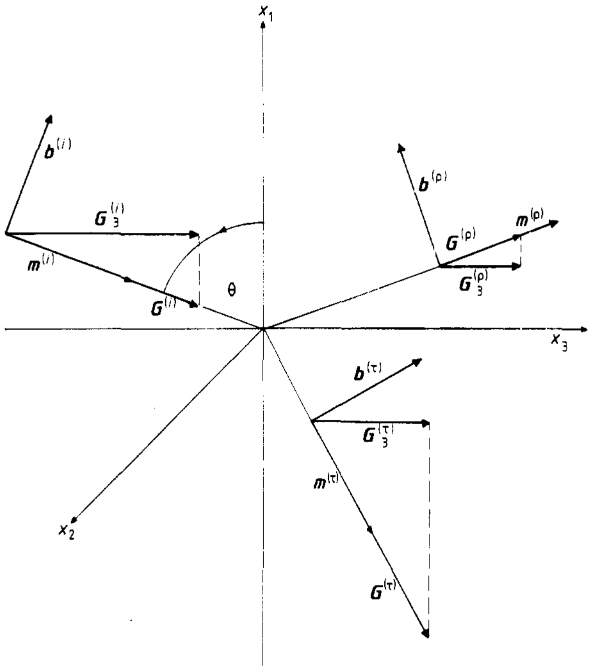

Figure 2. Reflection and transmission geometry. In the case of total reflection $\boldsymbol{G}^{(\tau)}(t_2) = 0$.

Denote the dimensions of the incident packet in $\boldsymbol{m}^{(i)}$, $\boldsymbol{a}$ and $\boldsymbol{b}^{(i)}$ directions at the instant of time t_1 by D_m, D_a and D_b. Suppose that $D_m \gg D_b$. As the incident wavepacket is quasimonochromatic, $D_{a,b} \gg \lambda$ where λ is the mean vacuum wavelength of the packet. Since the packets are far enough from the interface at the instants t_1 and t_2, the condition $t_2 - t_1 > n^{(1)} D_m$ should be fulfilled. We neglect the diffusion of the packets during the time interval $t_2 - t_1$, for which the condition

$$\lambda(t_2 - t_1) \ll D_{a,b}^2$$

should be fulfilled.

The relations between the quantities characterising the incident, reflected and transmitted quasimonochromatic wavepackets are approximately (in the zeroth order in $\lambda D_{a,b}^{-1}$) the same as those between the quantities characterising the plane electromagnetic waves with the wavevectors $\boldsymbol{k}^{(j)} = 2\pi n^{(j)}\lambda^{-1}\boldsymbol{m}^{(j)}$. The vectors $\boldsymbol{k}^{(\rho)}$ and $\boldsymbol{k}^{(\tau)}$ are connected with $\boldsymbol{k}^{(i)}$ by Snell's laws.

The process of the reflection and transmission of the quasimonochromatic wavepacket on the plane dielectric interface is a special case of the process considered in the previous section. Hence, the conservation laws (7*a*, *b*) and (14) are valid for this process. Relations (12*a*, *b*), (20) and (22*a*) are also valid, but they can be specified in this case as follows.

Denote the TS of the reflected and transmitted wavepackets by $h^{(\rho)}$ and $h^{(\tau)}$. Let us define $h^{(\rho)}$ and $h^{(\tau)}$ as the differences of x_2 coordinates of the centres of gravity of the reflected and transmitted wavepackets at the instant of time t_2 and the centre of gravity of the incident packet at the instant t_1 (figure 3):

$$h^{(\rho,\tau)} = X_2^{(\rho,\tau)}(t_2) - X_2^{(i)}(t_1). \tag{24}$$

One can obtain from (19), (24) and (A10*d*) that

$$h = Rh^{(\rho)} + Th^{(\tau)} \tag{25}$$

where $R = W^{(\rho)}(t_2)W^{-1}$, $T = W^{(\tau)}(t_2)W^{-1}$ and $R + T = 1$ (see (A10*a*)). In the zeroth order in $\lambda D_{a,b}^{-1}$ the quantities R and T are the reflectivity and transmissivity of the plane electromagnetic wave with the polarisation vector $\boldsymbol{e}^{(i)}$ (A12*a*) and the wavevector $\boldsymbol{k}^{(i)}$ (Born and Wolf 1964).

Consider relation (20). As $W^{(i)}(t_1) = W$ and $G_3^{(i)}(t_1) = G_3$, then using (A17) we get

$$G_3 = n^{(1)} W \sin\theta. \tag{26}$$

At the instant of time t_2 the normal component of the intrinsic angular momentum of the electromagnetic field is equal to the sum of the normal components of the intrinsic angular momenta of the reflected and transmitted packets (A19) (note that such

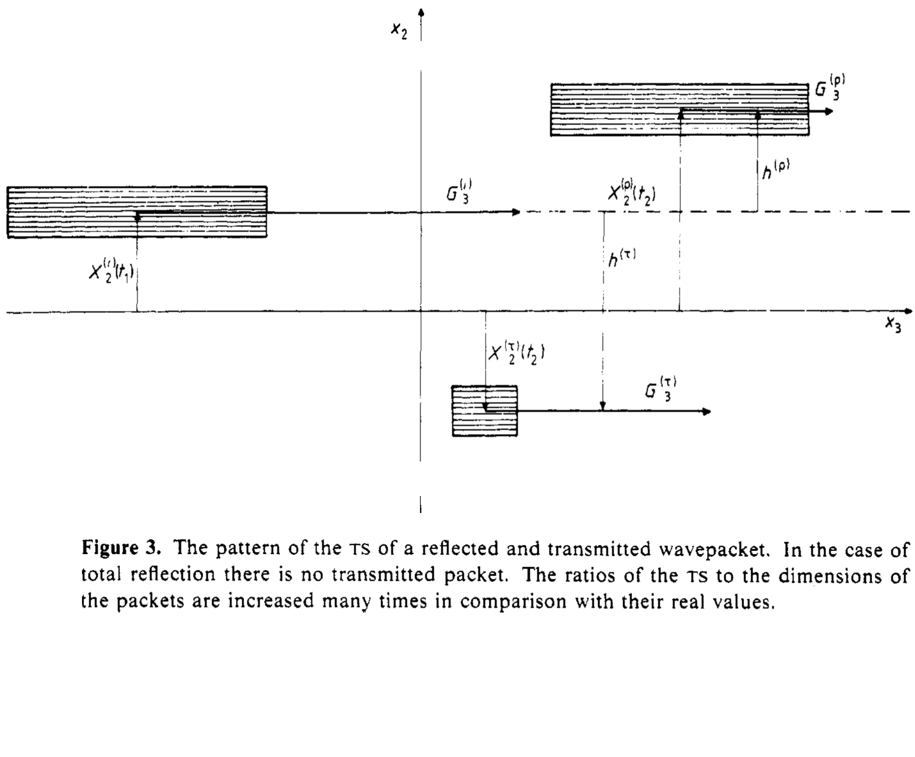

Figure 3. The pattern of the TS of a reflected and transmitted wavepacket. In the case of total reflection there is no transmitted packet. The ratios of the TS to the dimensions of the packets are increased many times in comparison with their real values.

affirmation is correct only for the Minkowski angular momenta). By means of (21), (25), (26) and (A19) relation (20) takes the form

$$Rh^{(\rho)}+Th^{(\tau)}=\frac{\lambda}{2\pi n^{(1)}\sin\theta}F_1 \tag{20a}$$

where

$$\boldsymbol{F}=-\frac{2\pi}{\lambda W}(\boldsymbol{I}^{(\rho)}(t_2)+\boldsymbol{I}^{(\tau)}(t_2)-\boldsymbol{I}^{(i)}(t_1)). \tag{27}$$

The scheme of the calculation of angular momentum of the electromagnetic field in vacuum is well known (see e.g. Akhiezer and Berestetskii 1965). In the appendix, the calculation of $\boldsymbol{I}^{(j)}(t_j)$ is performed according to this scheme. A substitution of (A9a) into (27) yields

$$\boldsymbol{F}=\mathrm{i}(R\boldsymbol{e}^{(\rho)}\times\boldsymbol{e}^{(\rho)*}+T\boldsymbol{e}^{(\tau)}\times\boldsymbol{e}^{(\tau)*}-\boldsymbol{e}^{(i)}\times\boldsymbol{e}^{(i)*}). \tag{27a}$$

Substituting (27a) into (20a) we obtain the relation between the TS of the centre of gravity of the reflected and transmitted wavepackets and the change of the polarisation vector on reflection and transmission of the packet.

The vector $\boldsymbol{F}$ can also be expressed through the components of the electric field vector of the incident packet. Let us present the polarisation vector of the jth quasimonochromatic wavepacket as

$$\boldsymbol{e}^{(j)}=\frac{A^{(j)}\boldsymbol{a}+B^{(j)}\boldsymbol{b}^{(j)}}{|A^{(j)}|^2+|B^{(j)}|^2} \tag{28}$$

where $A^{(j)}$ and $B^{(j)}$ are the transversal and planar components of the electric field vector of the jth packet (cf Fedorov 1977, Fedoseyev 1985). Substituting (28) into (27a) and using Fresnel's laws one gets after some algebra

$$F_1=-\frac{1-\varepsilon}{\varepsilon^{1/2}}\cos\theta\frac{\mathrm{Im}(\tau_\perp^*\tau_\parallel A^{(i)*}B^{(i)})}{|A^{(i)}|^2+|B^{(i)}|^2} \tag{27b}$$

where $\tau_\perp$ and $\tau_\parallel$ are the field transmission coefficients of the plane electromagnetic wave with the wavevector $\boldsymbol{k}^{(i)}$ and with the polarisation denoted by lower indices (Born and Wolf 1964).

Consider relation (22a). It can be compared with the result obtained by means of the energy-flux method. For that let D_a tend to infinity, with the assumption that the electromagnetic field inside the incident packet is independent of x_2. Denote

$$S_2(t)=D_aP_2(t) \tag{29}$$

and similarly

$$S_2^{(1,2)}(t)=D_aP_2^{(1,2)}(t). \tag{29a}$$

$P_2(t)$ is the TPF, i.e. the power transported through the $x_2=$ constant plane, $P_2^{(1)}(t)$ and $P_2^{(2)}(t)$ being the TPF in the first and the second media. Substituting (29) and (29a) into (12a, b) one gets

$$P_2^{(1)}(t)=(1-\varepsilon^{-1})^{-1}P_2(t) \tag{30a}$$

$$P_2^{(2)}(t)=(1-\varepsilon)^{-1}P_2(t). \tag{30b}$$

The incident packet contacts the interface during the time interval

$$\Delta t\simeq n^{(1)}D_m \tag{31}$$

(we take into account that $D_m \gg D_b$). Assume that the incident packet at the instant of time t_1 is nearly homogeneous along the coordinate $x_m^{(i)} = \boldsymbol{m}^{(i)} \cdot \boldsymbol{x}$ in its main part. Then taking into account (29) and (31), one gets

$$\int_{t_1}^{t_2} S_2(t)\,\mathrm{d}t \simeq n^{(1)} D_m D_a P_2 \tag{32}$$

where P_2 is the mean value of the TPF during the time interval, when the incident packet gets into contact with the interface. $P_2 \simeq P_2(t_0)$ where t_0 is the instant of time at which the middle part of the incident packet gets into contact with the interface (figure 1(*b*)). Substituting (25) into the left-hand side of (22*a*) and (32) into its right-hand side, we obtain

$$Rh^{(\rho)} + Th^{(\tau)} \simeq n^{(1)} v^{-1} P_2 \tag{22b}$$

where $v = (D_a D_m)^{-1} W$ is the electromagnetic energy of a part of the incident packet of unit dimensions in the $\boldsymbol{m}^{(i)}$ and $\boldsymbol{a}$ directions.

Finally, let us note that the TS of the reflected and transmitted time-restricted wavepackets coincide in the zeroth approximation with the TS of the reflected and transmitted wavebeams. It is easy to be convinced of this if, in the incident wavebeam with the mean wavevector $\boldsymbol{k}^{(i)}$, one mentally distinguishes a part of the length D_m which at the instant of time t_1 occurs in the same position as the wavepacket (figure 1(*a*)) and traces its motion during the time interval $t_2 - t_1$ (figure 1(*b*, *c*)).

4. Conclusions and discussion of results

Our reasoning proceeds from the fact that electromagnetic energy, the components of the Minkowski momentum parallel to the interface and the component of the Minkowski angular momentum normal to the interface are the constants of motion of the electromagnetic field in a space filled with two isotropic dielectric media separated by a plane interface.

On the basis of these constants of motion we have established the following relations between the phenomena associated with the process of reflection and transmission of electromagnetic waves on the plane interface.

(1) The relation between the transverse Abraham momenta in the first and second media (equations (11) and (12*a*, *b*)) or the relation between the TPF in the first and second media, if the incident wavepacket is nearly homogeneous in the transverse direction (30*a*, *b*).

(2) Two relations involving the TS of the centre of gravity of the reflected and transmitted quasimonochromatic wavepackets h (25) as follows.

(i) The relation between h and $\Delta I_1 = I_1^{(\rho)}(t_2) + I_1^{(\tau)}(t_2) - I_1^{(i)}(t_1)$ (equations (20*a*) and (27)), where $I_1^{(\rho)}(t_2)$ and $I_1^{(\tau)}(t_2)$ are the normal components of the Minkowski angular momenta of the reflected and transmitted wavepackets at the final instant of time t_2 and $I_1^{(i)}(t_1)$ is that of the incident wavepacket at the initial instant of time t_1. This relation can also be transformed to that between h and the change in the polarisation vector on reflection and transmission (equations (20*a*) and (27*a*)).

(ii) The relation between h and the integral transverse Abraham momentum (22*a*). If the incident wavepacket is long and nearly homogeneous, this relation goes over into that between h and the TPF (22*b*).

Earlier, the relation between the TS and the TPF was obtained by means of the energy-flux method for the case of total reflection (Kristoffel 1956, Imbert 1972, Fedorov 1977). Denote by $h^{(p)}_{EFM}$ the TS of the totally reflected light beam obtained by means of this method. The formula for $h^{(p)}_{EFM}$ can be found in equation (12) of Imbert (1972) or equation (5) of Fedorov (1977), for instance. Applying these equations for a wavebeam of finite thickness and keeping in mind that the power flux on the strip of the interface of a unit dimension in $\boldsymbol{a}$ direction is equal to $(n^{(1)})^{-1}v$, one can express the quantity $h^{(p)}_{EFM}$ as $h^{(p)}_{EFM}=n^{(1)}v^{-1}P_2^{(2)}$.

Two assumptions are made in developing the energy-flux method. First, the TPF is assumed to be associated only with the inhomogeneous wave field, i.e. within this method $P_2\equiv P_2^{(2)}$. Second, in this method an assumption known as the 'energy-flux-conservation argument' (Imbert 1972) is used, i.e. it is assumed that on total reflection the incident light 'dives' into a less dense medium at some place of the interface re-emerging in a denser medium in another place. This assumption seems to be incorrect, as discussed by the author in connection with the longitudinal motion of electromagnetic energy (Fedoseyev and Adamson 1981, Fedoseyev 1986). Nevertheless, owing to its applications, the ratio $h^{(p)}_{EFM}(P_2^{(2)})^{-1}$ proves equal to the ratio $h^{(p)}P_2^{-1}$ which is obtained from (22*b*) for total reflection when $T=0$ and $h=h^{(p)}$. (Note that the ratios (22*a*, *b*) are obtained without any assumption about the process of motion of energy on reflection and transmission of electromagnetic waves.) So, in the case of total reflection, the reason for the numerical difference between the values $h^{(p)}$ and $h^{(p)}_{EFM}$ is caused by the difference between P_2 and $P_2^{(2)}$.

Consider the TPF phenomenon. Based on conservation law (9*a*), one can conclude that the TPF cannot be confined to the inhomogeneous wave field. Indeed, if on total reflection the TPF equal to $P_2^{(2)}(t)$, associated with the evanescent wave field, exists in a less dense medium then, in accordance with (30*a*, *b*), the TPF equal to $P_2^{(1)}(t)=-\varepsilon P_2^{(2)}(t)$ should exist in a denser medium, where there are no inhomogeneous waves. This conclusion contradicts the traditional point of view on the TPF phenomenon, being in agreement with the author's recent paper (Fedoseyev 1987), which shows that in a general case the TPF is associated with the fields of both inhomogeneous and homogeneous waves, and that in a denser medium the interference TPF also exists.

De Beauregard *et al* (1971) regarded the non-zero value of $P_2^{(2)}(t)$ (or, more generally, $G_2^{(2)}(t)=\varepsilon^{(2)}S_2^{(2)}(t)$) on total reflection as an 'unsatisfactory fact' of the Minkowski energy-momentum tensor. One can see that their objection against this tensor is nullified if one takes into account the TPF in the second as well as in the first medium.

It is evident from the above discussion that the value of the TS of a totally reflected light beam obtained by the energy-flux method differs from that calculated by means of (22*b*) by the factor $(1-\varepsilon)^{-1}$ (see (30*b*)). It is just the difference between the values of $h^{(p)}$ obtained in the case of total reflection by the stationary-phase and energy-flux methods. Hence, one can explain this difference on the basis of the conservation law (9*a*).

Let us discuss relation (20). Its physical meaning is as follows. $M_1(t)$ is the constant of motion of the electromagnetic field. Hence, if the normal component of the intrinsic angular momentum of the electromagnetic field changes during reflection and transmission, then the field should be shifted in the transverse direction in order to evoke an opposite change of the normal component of the orbital angular momentum. Qualitatively, this interpretation of the TS phenomenon resembles de Beauregard's point of view on the TS as 'a translational inertial spin effect' (de Beauregard 1965).

However, it should be stressed that the intrinsic angular momentum in (20) and (27) is the Minkowski one.

Relation (20) gives a new method for calculating the quantity h (equations (20*a*) and (27*a*)). Expressing the polarisation vectors of wavepackets through the components of the electric field vectors (28), one obtains the result (equations (20*a*) and (27*b*)) which is convenient to compare with the previous calculations of $h^{(\rho)}$ and $h^{(\tau)}$. For total reflection the value $h^{(\rho)}$ calculated by means of (20*a*) and (27*b*) differs from the result of de Beauregard (1965) and coincides with that obtained by the stationary-phase method (Schilling 1965). For partial reflection the value h, obtained from (20*a*) and (27*b*), is in accordance with the previous calculations of $h^{(\rho)}$ (Schilling 1965, Fedoseyev 1985) and $h^{(\tau)}$ (Fedoseyev 1985).

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Appendix. Electromagnetic energy, centre of gravity, momentum and angular momentum of incident, reflected and transmitted quasimonochromatic wavepackets

$\boldsymbol{E}^{(j)}(\boldsymbol{x}, t_j)$ and $\boldsymbol{H}^{(j)}(\boldsymbol{x}, t_j)$ are the electric and magnetic field vectors of the jth wavepacket at the instant t_j.

The electromagnetic energy density $w^{(j)}(\boldsymbol{x}, t_j)$ and the Minkowski momentum density $\boldsymbol{g}^{(j)}(\boldsymbol{x}, t_j)$ of the jth wavepacket at instant t_j are

$$w^{(j)}(\boldsymbol{x}, t_j) = \frac{1}{8\pi}(\varepsilon^{(j)}|\boldsymbol{E}^{(j)}(\boldsymbol{x}, t_j)|^2 + |\boldsymbol{H}^{(j)}(\boldsymbol{x}, t_j)|^2) \tag{A1}$$

$$\boldsymbol{g}^{(j)}(\boldsymbol{x}, t_j) = \frac{\varepsilon^{(j)}}{4\pi}\boldsymbol{E}^{(j)}(\boldsymbol{x}, t_j) \times \boldsymbol{H}^{(j)}(\boldsymbol{x}, t_j) \tag{A2}$$

where $\varepsilon^{(i,\rho)} = \varepsilon^{(1)}$ and $\varepsilon^{(\tau)} = \varepsilon^{(2)}$.

The electromagnetic energy $W^{(j)}(t_j)$, the Minkowski momentum $\boldsymbol{G}^{(j)}(t_j)$, the Minkowski angular momentum $\boldsymbol{M}^{(j)}(t_j)$ and the radius vector of centre of gravity $\boldsymbol{X}^{(j)}(t_j)$ of the jth wavepacket at instant t_j are as follows:

$$W^{(j)}(t_j) = \int w^{(j)}(\boldsymbol{x}, t_j)\,\mathrm{d}\boldsymbol{x} \tag{A3}$$

$$\boldsymbol{G}^{(j)}(t_j) = \int \boldsymbol{g}^{(j)}(\boldsymbol{x}, t_j)\,\mathrm{d}\boldsymbol{x} \tag{A4}$$

$$\boldsymbol{M}^{(j)}(t_j) = \int \boldsymbol{x} \times \boldsymbol{g}^{(j)}(\boldsymbol{x}, t_j)\,\mathrm{d}\boldsymbol{x} \tag{A5}$$

$$\boldsymbol{X}^{(j)}(t_j) = (W^{(j)}(t_j))^{-1}\int \boldsymbol{x}w^{(j)}(\boldsymbol{x}, t_j)\,\mathrm{d}\boldsymbol{x}. \tag{A6}$$

The angular momentum of the jth wavepacket is the sum of the orbital and intrinsic angular momenta

$$\boldsymbol{M}^{(j)}(t_j) = \boldsymbol{L}^{(j)}(t_j) + \boldsymbol{I}^{(j)}(t_j) \tag{A7}$$

where

$$L^{(j)}(t_j) = X^{(j)}(t_j) \times G^{(j)}(t_j) \tag{A8}$$

$$I^{(j)}(t_j) = \int r^{(j)} \times g^{(j)}(x, t_j)\,\mathrm{d}x \tag{A9}$$

and $r^{(j)} = x - X^{(j)}(t_j)$.

The integration in (A3)-(A6) and (A9) is performed over the volume of the first medium for $j = i, \rho$ and over the volume of the second medium for $j = \tau$.

At instant t_2 the electromagnetic field of the reflected packet is the total electromagnetic field in the first medium; hence, combining (8) with (A3), (9) with (A4) and (13a) with (A5), we obtain

$$W(t_2) \equiv W = W^{(\rho)}(t_2) + W^{(\tau)}(t_2) \tag{A10a}$$

$$G(t_2) = G^{(\rho)}(t_2) + G^{(\tau)}(t_2) \tag{A10b}$$

$$M(t_2) = M^{(\rho)}(t_2) + M^{(\tau)}(t_2). \tag{A10c}$$

Using (16) and (A6) we get

$$X(t_2) = \frac{W^{(\rho)}(t_2)}{W} X^{(\rho)}(t_2) + \frac{W^{(\tau)}(t_2)}{W} X^{(\tau)}(t_2). \tag{A10d}$$

According to the definition of t_1 and t_2 the quantities $W^{(j)}(t_j)$, $G^{(j)}(t_j)$, $X^{(j)}(t_j)$ and $M^{(j)}(t_j)$ are approximately the electromagnetic energy, the Minkowski momentum, the radius vector of the centre of gravity and the angular momentum of the jth wavepacket moving in the homogeneous medium of the dielectric permittivity $\varepsilon^{(j)}$. When calculating these quantities, instead of integrating over the half-space in (A3)-(A6) and (A9), we can integrate over the total space with the dielectric permittivity $\varepsilon^{(j)}$.

Let us make the Fourier transform of the vectors $E^{(j)}(x, t_j)$ and $H^{(j)}(x, t_j)$:

$$E^{(j)}(x, t_j) = \frac{1}{(2\pi)^{3/2}} \operatorname{Re} \int E^{(j)}(k) \exp(-\mathrm{i}k \cdot r^{(j)})\,\mathrm{d}k \tag{A11a}$$

$$H^{(j)}(x, t_j) = \frac{1}{(2\pi)^{3/2}} \operatorname{Re} \int H^{(j)}(k) \exp(-\mathrm{i}k \cdot r^{(j)})\,\mathrm{d}k. \tag{A11b}$$

Note that at an arbitrary instant t the time dependence of the vectors $E^{(j)}(k)$ and $H^{(j)}(k)$ is given by a factor $\exp(\mathrm{i}\omega(k)t)$, where $\omega^2(k) = (\varepsilon^{(j)})^{-1}k^2$. The scalar products $E^{(j)}(k) \cdot E^{(j)}(k')$ and $H^{(j)}(k) \cdot H^{(j)}(k')$ as well as the complex conjugates and the corresponding vector products oscillate at the frequency $\sim 2\omega_0$, where $\omega_0 = 2\pi\lambda^{-1}$. In further calculations these products will be omitted.

Let us introduce the following notation:

$$e^{(j)}(k) = |E^{(j)}(k)|^{-1} E^{(j)}(k). \tag{A12}$$

Here $e^{(j)}(k)$ is the unit polarisation vector of the plane electromagnetic wave with the wavevector k present in the jth packet,

$$e^{(j)} \equiv e^{(j)}(k^{(j)}) \tag{A12a}$$

being a unit polarisation vector of the jth wavepacket.

The vectors $\boldsymbol{k}$, $\boldsymbol{E}^{(j)}(\boldsymbol{k})$ and $\boldsymbol{H}^{(j)}(\boldsymbol{k})$ satisfy the relations

$$\boldsymbol{k}\cdot\boldsymbol{E}^{(j)}(\boldsymbol{k})=0 \tag{A13}$$

$$\boldsymbol{H}^{(j)}(\boldsymbol{k})=(n^{(j)}/k)\boldsymbol{k}\times\boldsymbol{E}^{(j)}(\boldsymbol{k}) \tag{A14}$$

which follow from the Maxwell equations, $n^{(j)}=(\varepsilon^{(j)})^{1/2}$.

When calculating the quantities $\boldsymbol{G}^{(j)}(t_j)$ and $\boldsymbol{I}^{(j)}(t_j)$, the following function will be used:

$$\boldsymbol{g}^{(j)}(\boldsymbol{k},\boldsymbol{k}')\equiv\frac{\varepsilon^{(j)}}{16\pi}(\boldsymbol{E}^{(j)}(\boldsymbol{k}')\times\boldsymbol{H}^{(j)*}(\boldsymbol{k})+\boldsymbol{E}^{(j)*}(\boldsymbol{k})\times\boldsymbol{H}^{(j)}(\boldsymbol{k}')). \tag{A15}$$

Substituting (A14) into the right-hand side of (A15) and performing some transformations of vector products with the use of (A13), one can transform this function into the form

$$\boldsymbol{g}^{(j)}(\boldsymbol{k},\boldsymbol{k}')=\frac{(n^{(j)})^3}{16\pi}\left\{\left(\frac{\boldsymbol{k}}{k}+\frac{\boldsymbol{k}'}{k'}\right)\boldsymbol{E}^{(j)}(\boldsymbol{k}')\cdot\boldsymbol{E}^{(j)*}(\boldsymbol{k})\right.$$
$$\left.+\left(\frac{\boldsymbol{k}}{k}-\frac{\boldsymbol{k}'}{k'}\right)\times\boldsymbol{E}^{(j)}(\boldsymbol{k}')\times\boldsymbol{E}^{(j)*}(\boldsymbol{k})\right\}. \tag{A15a}$$

Substituting (A11a, b) into (A1), and the result into (A3), and using (A13) and (A14) we get

$$W^{(j)}(t_j)=\frac{\varepsilon^{(j)}}{8\pi}\int|\boldsymbol{E}^{(j)}(\boldsymbol{k})|^2\,\mathrm{d}\boldsymbol{k}. \tag{A3a}$$

Analogously, substituting (A11a, b) into (A2), and the result into (A4) and using (A15a), we get

$$\boldsymbol{G}^{(j)}(t_j)=\frac{(n^{(j)})^3}{8\pi}\int\frac{\boldsymbol{k}}{k}|\boldsymbol{E}^{(j)}(\boldsymbol{k})|^2\,\mathrm{d}\boldsymbol{k}. \tag{A4a}$$

For every index j the wavevector $\boldsymbol{k}$ can be presented as $\boldsymbol{k}=\kappa_m^{(j)}\boldsymbol{m}^{(j)}+\boldsymbol{\kappa}_\perp^{(j)}$, where $\boldsymbol{\kappa}_\perp^{(j)}=\kappa_a^{(j)}\boldsymbol{a}+\kappa_b^{(j)}\boldsymbol{b}^{(j)}$ is the component of the wavevector $\boldsymbol{k}$ perpendicular to $\boldsymbol{m}^{(j)}$. Let us denote the half-widths of the functions $\boldsymbol{E}^{(j)}(\boldsymbol{k})$ and $\boldsymbol{H}^{(j)}(\boldsymbol{k})$ in the $\boldsymbol{m}^{(j)}$, $\boldsymbol{a}$ and $\boldsymbol{b}^{(j)}$ directions by $\Delta\kappa_{m,a,b}^{(j)}$. The half-widths are connected with the dimensions of the packets as follows: $\Delta\kappa_m^{(j)}\sim D_m^{-1}$, $\Delta\kappa_a^{(j)}\sim D_a^{-1}$, $\Delta\kappa_b^{(j)}\sim D_b^{-1}$. As $\Delta\kappa_{m,a,b}^{(j)}\ll k^{(j)}$, the factor $k^{-1}\boldsymbol{k}$ in (A4a) can be expanded into a series, while in the first order in $\lambda D_{a,b}^{-1}$ we have

$$\frac{\boldsymbol{k}}{k}=\boldsymbol{m}^{(j)}+\frac{\lambda\boldsymbol{\kappa}_\perp^{(j)}}{2\pi n^{(j)}}. \tag{A16}$$

The substitution of (A16) into (A4a) yields in the first approximation

$$\boldsymbol{G}^{(j)}(t_j)=n^{(j)}W^{(j)}(t_j)\boldsymbol{m}^{(j)}. \tag{A4b}$$

By the use of Snell's laws (A4b) is transformed into the relation

$$\frac{G_3^{(j)}(t_j)}{W^{(j)}(t_j)}=n^{(1)}\sin\theta. \tag{A17}$$

Substituting (A10b, d) into (15) and using (9a, b) and (A17), we get

$$L_1(t_2)=L_1^{(\rho)}(t_2)+L_1^{(\tau)}(t_2). \tag{A18}$$

Substituting (13*b*) into the left-hand side of (A10*c*) and (A7) into its right-hand side and using (A18) one gets

$$I_1(t_2)=I_1^{(\rho)}(t_2)+I_1^{(\tau)}(t_2). \tag{A19}$$

Let us calculate the intrinsic angular momentum of the jth wavepacket. Substituting (A11*a*, *b*) into (A2), and the result into (A9) and using the definition of the quantity $\boldsymbol{g}^{(j)}(\boldsymbol{k},\boldsymbol{k}')$ (A15), we obtain

$$\boldsymbol{I}^{(j)}(t_j)=(2\pi)^{-3}\iiint \boldsymbol{r}^{(j)}\times\boldsymbol{g}^{(j)}(\boldsymbol{k},\boldsymbol{k}')\exp[\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}^{(j)}]\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}'\,\mathrm{d}\boldsymbol{x}. \tag{A20}$$

On calculating $\boldsymbol{I}^{(j)}(t_j)$ the terms small in comparison with $\lambda W^{(j)}(t_j)$ will be omitted. In this approximation the expansion (A16) can be used in (A15*a*). We can also neglect the dependence of the polarisation vector $\boldsymbol{e}^{(j)}(\boldsymbol{k})$ on $\boldsymbol{k}$ in the terms containing the factor $\lambda\boldsymbol{\kappa}_\perp^{(j)}$. Taking that into account, the function $\boldsymbol{g}^{(j)}(\boldsymbol{k},\boldsymbol{k}')$ can be divided into three parts:

$$\boldsymbol{g}^{(j)}(\boldsymbol{k},\boldsymbol{k}')\simeq\boldsymbol{g}_\parallel^{(j)}(\boldsymbol{k},\boldsymbol{k}')+\boldsymbol{g}_\perp^{(j1)}(\boldsymbol{k},\boldsymbol{k}')+\boldsymbol{g}_\perp^{(j2)}(\boldsymbol{k},\boldsymbol{k}') \tag{A15b}$$

where

$$\boldsymbol{g}_\parallel^{(j)}(\boldsymbol{k},\boldsymbol{k}')=\frac{(n^{(j)})^3}{8\pi}(\boldsymbol{E}^{(j)}(\boldsymbol{k}')\cdot\boldsymbol{E}^{(j)*}(\boldsymbol{k}))\boldsymbol{m}^{(j)} \tag{A21}$$

$$\boldsymbol{g}_\perp^{(j1)}(\boldsymbol{k},\boldsymbol{k}')=\frac{\varepsilon^{(j)}\lambda}{32\pi^2}|\boldsymbol{E}^{(j)}(\boldsymbol{k})||\boldsymbol{E}^{(j)}(\boldsymbol{k}')|(\boldsymbol{\kappa}_\perp^{(j)}+\boldsymbol{\kappa}_\perp^{\prime(j)}) \tag{A22}$$

$$\boldsymbol{g}_\perp^{(j2)}(\boldsymbol{k},\boldsymbol{k}')=\frac{\varepsilon^{(j)}\lambda}{32\pi^2}|\boldsymbol{E}^{(j)}(\boldsymbol{k})||\boldsymbol{E}^{(j)}(\boldsymbol{k}')|(\boldsymbol{\kappa}_\perp^{(j)}-\boldsymbol{\kappa}_\perp^{\prime(j)})\times\boldsymbol{e}^{(j)}\times\boldsymbol{e}^{(j)*}. \tag{A23}$$

The integral in the right-hand side of (A20) containing the function $\boldsymbol{g}_\parallel^{(j)}(\boldsymbol{k},\boldsymbol{k}')$ is equal to zero. One will be convinced of that if one first performs integration over $\boldsymbol{k}$ and $\boldsymbol{k}'$ and then takes into account that (see (A6))

$$\int \boldsymbol{r}^{(j)}w^{(j)}(\boldsymbol{x},t_j)\,\mathrm{d}\boldsymbol{x}=0.$$

The integral containing the function $\boldsymbol{g}_\perp^{(j1)}(\boldsymbol{k},\boldsymbol{k}')$ is also equal to zero as its sign changes on the following simultaneous substitution of variables in (A20):

$$\boldsymbol{k}\to\boldsymbol{k}'\qquad \boldsymbol{k}'\to\boldsymbol{k}\qquad \boldsymbol{x}\to-\boldsymbol{x}+2\boldsymbol{X}^{(j)}(t_j).$$

In order to calculate the integral in the right-hand side of (A20), containing the function $\boldsymbol{g}_\perp^{(j2)}(\boldsymbol{k},\boldsymbol{k}')$, let us use the identity

$$\boldsymbol{r}^{(j)}\exp[\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}^{(j)}]=-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{k}}\exp[\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}^{(j)}]$$

and first perform integration over $\boldsymbol{k}$ by parts and then integration over $\boldsymbol{x}$. Further calculations of this integral are straightforward. Using (A3*a*), we finally obtain in the first approximation

$$\boldsymbol{I}^{(j)}(t_j)=-\mathrm{i}\frac{\lambda}{2\pi}W^{(j)}(t_j)\boldsymbol{e}^{(j)}\times\boldsymbol{e}^{(j)*}. \tag{A9b}$$

References

Akhiezer A I and Berestetskii V B 1965 *Quantum Electrodynamics* (New York: Interscience) ch 3
Ashby N and Miller S C Jr 1973 *Phys. Rev.* D 7 2383
—— 1977 *J. Opt. Soc. Am.* **67** 448
de Beauregard O C 1965 *Phys. Rev.* **139** B1443
de Beauregard O C, Imbert C and Ricard J 1971 *Int. J. Theor. Phys.* **4** 125
de Beauregard O C, Imbert C and Levi Y 1977 *Phys. Rev.* D **15** 3553
Born M and Wolf E 1964 *Principles of Optics* (Oxford: Pergamon)
Boulware D G 1973 *Phys. Rev.* D 7 2375
Ginsburg V L 1973 *Usp. Fiz. Nauk* **110** 309
Fedorov F I 1955 *Dokl. Akad. Nauk* **105** 465
—— 1977 *Zh. Prikl. Spektrosk.* **27** 580
Fedoseyev V G 1985 *Opt. Spektrosk.* **58** 491
—— 1986 *J. Opt. Soc. Am.* A **3** 826
—— 1987 *Opt. Spektrosk.* **62** 119
Fedoseyev V G and Adamson P V 1981 *Preprint* F-16, Estonian SSR, Academy of Sciences, Tartu (in Russian)
Imbert C 1972 *Phys. Rev.* D **5** 787
Kristoffel N N 1956 *Sci. Acta Tartu State Univ.* **42** 94
Møller C 1972 *The Theory of Relativity* (Oxford: Clarendon)
Pun'ko N N and Filippov V V 1984 *Zh. Eksp. Teor. Fiz. Pis. Red.* **39** 18
—— 1985 *Opt. Spektrosk.* **58** 125
Schilling H 1965 *Ann. Phys., NY* **16** 122
Wiegrefe A 1914 *Ann. Phys., Lpz.* **45** 465